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Rigidity of GKM-graphs via 1-ideals

YAMANAKA Hitoshi

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1 Introduction

The aim of the present paper is to give a proof of equivariant cohomological rigidity of GKM graphs [1].

We first recall some notations following [1]. Let $\mathcal{G}$ and $\mathcal{G}'$ be two abstract GKM-graphs of type (r, n) and (r, n') respectively. We denote by $H_T^*(\mathcal{G})$ and $H_T^*(\mathcal{G}')$ the corresponding graph equivariant cohomology of $\mathcal{G}$ and $\mathcal{G}'$ respectively. There exists the notion of an isomorphism $\varphi : \mathcal{G}' \to \mathcal{G}$.

Our main theorem in this paper can be stated as follows:

Theorem 1.1. (Graph equivariant cohomological rigidity for GKM-graphs) $H_T^*(\mathcal{G})$ and $H_T^*(\mathcal{G}')$ are isomorphic as graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebras if and only if $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as GKM-graphs.

We will introduce the notion of a 1-ideal I_{pq} associated with two distinct vertices p, q of $\mathcal{G}$. The set of 1-ideals gives an algebraic analogue of the 1-skeleton of a GKM-manifold and well-behaves under any isomorphism $H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$ of graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebras (see Corollary 6.2). The proof of Theorem 1.1 is reduced to the classification of isomorphic types of 1-ideals.

Notation. Throughout this paper, we fix positive integers r, n, n' satisfying $r \leq n, n'$. We denoted by $|S|$ the number of elements in a finite set S. We regard the polynomial ring $\mathbb{Z}[x_1, \ldots, x_r]$ as a graded ring with deg $x_i = 2$. The module $\mathbb{Z}[x_1, \ldots, x_r]_2$ is defined to be the degree 2 component of the polynomial ring. In other words $\mathbb{Z}[x_1, \ldots, x_r]_2 = \{a_1x_1 + \cdots + a_rx_r | a_1, \ldots, a_r \in \mathbb{Z}\}$. For two polynomials $P, Q \in \mathbb{Z}[x_1, \ldots, x_r]$, we write $P|Q$ if $Q = RP$ for some $R \in \mathbb{Z}[x_1, \ldots, x_r]$.

We use the same notions and terminologies as in [1].

2 Equivariant Thom classes for fixed points

Following Guillemin-Zara [3], we introduce the notion of the equivariant Thom class corresponding to a vertex of $\mathcal{G}$.

Definition 2.1. For any $p \in \mathcal{V}$, we define a map τ_p: $\mathcal{V} \to \mathbb{Z}[x_1, \ldots, x_r]$ by

$$\tau_p(q) := \begin{cases} \prod_{e \in \mathcal{E}_p} \alpha(e) & (q = p) \\ 0 & (q \neq p). \end{cases}$$

The map τ_p is called the **equivariant Thom class** associated with p.

Note that equivariant Thom classes are clearly in $H_T^{2n}(\mathcal{G})$.

Definition 2.2. Let A be a commutative ring and B be an A-algebra. Then an element $b_0 \in B$ is said to be **maximal**, if the sub A-module Ab_0 generated by b_0 is maximal in the set $\{Ab | b \in B\}$ with respect to inclusion.

The following lemma is clear:

Lemma 2.3. Any equivariant Thom class is a maximal element of the $\mathbb{Z}[x_1, \ldots, x_r]$-algebra $H_T^*(\mathcal{G})$.

Proposition 2.4. Assume that N-elements $f_1, \ldots, f_N$ of $H_T^*(\mathcal{G})$ satisfy the following three conditions:
(i) Each f_i is maximal.
(ii) $f_i^2 = P_i f_i$ for some $P_i \in \mathbb{Z}[x_1, \ldots, x_r]$.
(iii) $f_i f_j = 0$ for all distinct i, j.
Then we have $N \leq |\mathcal{V}|$. The equality holds if and only if $f_i = \varepsilon_i \tau_{p_i}$ for some $\varepsilon_i \in \{\pm 1\}$ and $p_i \in \mathcal{V}$.

Proof. The condition (ii) implies that any component of f_i is 0 or P_i. In addition every f_i has at least one non-zero component by the condition (i). Moreover, by the condition (iii), there does not exist a vertex p such that $f_i(p) \neq 0$ and $f_j(p) \neq 0$ for some distinct i, j. Thus the pigeon hole principle implies the desired inequality. The rest follows from the maximality of equivariant Thom classes (Lemma 2.3) and the condition (i). □

Corollary 2.5. We have $|\mathcal{V}| = |\mathcal{V}'|$ if $H_T^*(\mathcal{G})$ and $H_T^*(\mathcal{G}')$ are isomorphic as $\mathbb{Z}[x_1, \ldots, x_r]$-algebras.

Proof. By Proposition 2.4, we have

$$|\mathcal{V}| = \sup\left\{ N \in \mathbb{N} \,\middle|\, \exists f_1, \ldots, f_N \text{ satisfying (i),(ii),(iii) in Proposition 2.4} \right\}.$$

This completes the proof since the right hand side is an invariant of $\mathbb{Z}[x_1, \ldots, x_r]$-algebras. □

The following theorem plays a crucial role in the rest of this paper:

Theorem 2.6. For any graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebra isomorphism φ: $H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$, there exists a unique bijection $\varphi_{\mathcal{V}}$: $\mathcal{V}' \to \mathcal{V}$ such that $\varphi(\tau_{\varphi_{\mathcal{V}}(p')}) = \varepsilon_{p'} \tau_{p'}$ for some $\varepsilon_{p'} \in \{\pm 1\}$.

Proof. By Corollary 2.5 the set $\{\varphi(\tau_p)\}_{p \in \mathcal{V}}$ attains the equality of the inequality in Proposition 2.4. □

Corollary 2.7. We have $n = n'$ if $H_T^*(\mathcal{G})$ and $H_T^*(\mathcal{G}')$ are isomorphic as graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebras.
Proof. This immediately follows from Theorem 2.6.

3 0- and 1-ideals

In this section we introduce the notion of 0- and 1-ideals in $H_T^*(\mathcal{G})$. We also explain our strategy for proving our main theorem.

Definition 3.1. Let p, q be distinct vertices of $\mathcal{G}$. We set

$$I_p := \{f \in H_T^*(\mathcal{G}) \mid f(r) = 0 \text{ for all } r \in \mathcal{V} \setminus \{p\}\},$$
$$I_{pq} := \{f \in H_T^*(\mathcal{G}) \mid f(r) = 0 \text{ for all } r \in \mathcal{V} \setminus \{p, q\}\}.$$

We call I_p the **0-ideal** associated with p and I_{pq} the **1-ideal** associated with p, q.

Remark 3.2. 0- and 1-ideals are algebraic counterparts of 0- and 1-skeletons of GKM-manifolds, respectively.

Lemma 3.3. The 0-ideal I_p is generated by the equivariant Thom class τ_p.

Lemma 3.4. If p and q are not adjacent, the 1-ideal I_{pq} is generated by the equivariant Thom classes τ_p, τ_q. In particular, I_{pq} does not contain a non-zero homogeneous element whose degree is less than or equal to $2(n-1)$.

4 Structure of 1-ideals

In this section we reveal the structure of the 1-ideal I_{pq} as a $\mathbb{Z}[x_1, \ldots, x_r]$-module.

Throughout this section, we fix adjacent vertices p, q of $\mathcal{G}$. We then introduce the following polynomials:

$$P := \prod_{e \in \mathcal{E}_p \setminus \mathcal{E}_{pq}} \alpha(e), \qquad M_p := \prod_{e \in \mathcal{E}_{pq}} \alpha(e),$$

$$Q := \prod_{e \in \mathcal{E}_q \setminus \mathcal{E}_{pq}} \alpha(e), \qquad M_q := \prod_{e \in \mathcal{E}_{qp}} \alpha(e)$$

("M" stands for "Middle of p and q").

Remark 4.1. (i) $\deg P = \deg Q = 2n - \deg M_p = 2n - \deg M_q$.
(ii) We have $M_q = \varepsilon M_p$ for some $\varepsilon \in \{\pm 1\}$.
For any $e \in \mathcal{E}_{pq}$ we set

$$C(e) := |\{e' \in \mathcal{E}_{pq} | e' \neq e, \alpha(\overline{e'}) = -\alpha(e')\}|$$

("C" stands for "Change of sign").

The following is a key lemma in this section. For its proof, the existence of a parallel transport on $\mathcal{G}$ is essential:

Lemma 4.2. The polynomial $P - (-1)^{C(e)} Q$ is divisible by $\alpha(e)$ for any $e \in \mathcal{E}_p$.

Proof. Let $\mathcal{P}$ be a parallel transport over $\mathcal{G}$. By the condition (iii) in Definition 2.2, there exist integers $\{d_{e,e'}\}_{e' \in \mathcal{E}_p}$ satisfying $\alpha(\mathcal{P}_e(e')) - \alpha(e') = d_{e,e'}\alpha(e)$ for any $e' \in \mathcal{E}_p$. Using this relation we have:

$$\begin{aligned}
P \cdot \prod_{e' \in \mathcal{E}_{pq}, e' \neq e} \alpha(e') &= \prod_{e' \in \mathcal{E}_p, e' \neq e} \alpha(e') \\
&= \prod_{e' \in \mathcal{E}_p, e' \neq e} (\alpha(\mathcal{P}_e(e')) - d_{e,e'}\alpha(e)) \\
&= \prod_{e' \in \mathcal{E}_p, e' \neq e} (\alpha(\mathcal{P}_e(e')) + (\text{terms divisible by } \alpha(e)) \\
&= \prod_{e'' \in \mathcal{E}_p, e'' \neq \overline{e}} \alpha(e'') + (\text{terms divisible by } \alpha(e)) \\
&= Q \cdot \prod_{e'' \in \mathcal{E}_{qp}, e'' \neq \overline{e}} \alpha(e'') + (\text{terms divisible by } \alpha(e)) \\
&= Q \cdot (-1)^{C(e)} \prod_{e' \in \mathcal{E}_{pq}, e' \neq \overline{e}} \alpha(e') + (\text{terms divisible by } \alpha(e)).
\end{aligned}$$

Thus

$$(P - (-1)^{C(e)} Q) \prod_{e' \in \mathcal{E}_{pq}, e' \neq \overline{e}} \alpha(e')$$

is divisible by $\alpha(e)$. Since $\alpha(e')$ ($e' \in \mathcal{E}_{pq}$, $e' \neq e$) and $\alpha(e)$ are coprime by the GKM-condition, the proof is now complete. □

Following Lemma 4.2 we set

$$E := \{e \in \mathcal{E}_{pq} | c(e) \text{ is even}\}, \quad O := \{e \in \mathcal{E}_{pq} | c(e) \text{ is odd}\}.$$

Notice that $\mathcal{E}_{pq}$ is the disjoint union of E and O. Lemma 4.2 immediately implies the following:

Corollary 4.3. The polynomials $P - Q$ and $P + Q$ are divisible by $\prod_{e\in E}\alpha(e)$ and $\prod_{e\in O}\alpha(e)$ respectively.

Corollary 4.4. We have

$$E = \{e \in \mathcal{E}_{pq} | P - Q \text{ is divisible by } \alpha(e)\},$$
$$O = \{e \in \mathcal{E}_{pq} | P + Q \text{ is divisible by } \alpha(e)\}.$$

Proof. In each case the inclusion $\subset$ is obvious by Lemma 4.2. Thus, it is enough to show that the intersection of the right hand sides is empty (note that $\mathcal{E}_{pq} = E \sqcup O$ as noticed above). Assume that $e \in \mathcal{E}_{pq}$ is contained in the intersection. Then both $P - Q$ and $P + Q$ are divisible by $\alpha(e)$. Thus $2P = (P + Q)+(P - Q)$ is so. This contradicts the GKM-condition. □

Following Corollary 4.4, we set

$$P_E := \prod_{e\in E} \mathrm{E}\alpha(e), \quad P_O := \prod_{e\in O} \in \mathrm{O}\alpha(e).$$

Remark 4.5. Since $\mathcal{E}_{pq} = E \sqcup O$, we have $M_p = P_E P_O$.

Notice that the 1-ideal I_{pq} is canonically identified with

$$I := \left\{ \begin{pmatrix} PA \\ QB \end{pmatrix} \in \mathbb{Z}[x_1, \ldots, x_r]^2 \middle| A, B \in \mathbb{Z}[x_1, \ldots, x_r], M_p|(PA - QB) \right\}.$$

In the rest of this section, we use this identification to simplify notation.

Lemma 4.6. There exists a bijection between the following sets:

$$S := \left\{ (A, B) \in \mathbb{Z}[x_1, \ldots, x_r]^2 \middle| M_p|(PA - QB) \right\},$$

$$S' := \left\{ (A', B') \in \mathbb{Z}[x_1, \ldots, x_r]^2 \middle| \ 2\,|(P_E A' + P_O B') \right\}.$$

Proof. We define two maps $f : S \to S'$ and $g : S' \to S$ by the following formula:

$$f(A, B) := \left(\frac{A - B}{P_E}, \frac{A + B}{P_O} \right),$$

$$g(A', B') := \left(\frac{P_E A' + P_O B'}{2}, \frac{-P_E A' + P_O B'}{2} \right).$$

It is easy to see that f and g are well-defined and are inverse to each other. □

By Lemma 4.6, to understand the structure of I, it is enough to understand the set S' of Lemma 4.6. Note that if (A', B') is in the set S', then $P_E A' + P_O B'$ is in $\langle P_E, P_O \rangle \cap \langle 2 \rangle$ where $\langle R \rangle$ is the ideal of $\mathbb{Z}[x_1, \ldots, x_r]$ generated by a subset R of $\mathbb{Z}[x_1, \ldots, x_r]$. Since $\mathbb{Z}[x_1, \ldots, x_r]$ is a Noetherian ring, the ideal $\langle P_E, P_O \rangle \cap \langle 2 \rangle$ is finitely generated. To obtain a finite generator of $\langle P_E, P_O \rangle \cap \langle 2 \rangle$ explicitly, we introduce two polynomials H_E and H_O as follows: we consider the natural ring homomorphism

$$\mathbb{Z}[x_1, \ldots, x_r] \to (\mathbb{Z}/2\mathbb{Z})[x_1, \ldots, x_r]$$

induced by mod 2 reduction of coefficients. We denote by $\overline{C}$ the image of $C \in \mathbb{Z}[x_1, \ldots, x_r]$ under this homomorphism. By the definition of primitive condition, we have $\overline{P_E} \neq 0$ and $\overline{P_O} \neq 0$. Thus we can consider the greatest common divisor $\gcd(\overline{P_E}, \overline{P_O})$. We choose two polynomials $H_E, H_O \in \mathbb{Z}[x_1, \ldots, x_r]$ so that

$$\overline{P_E} = \gcd(\overline{P_E}, \overline{P_O}) \cdot H_E, \quad \overline{P_O} = \gcd(\overline{P_E}, \overline{P_O}) \cdot \overline{H_O}.$$

Although the pair (H_E, H_O) is not unique, the following lemma holds for any choice of (H_E, H_O):

Lemma 4.7. We have $\langle P_E, P_O \rangle \cap \langle 2 \rangle = \langle 2P_E, 2P_O, P_E H_O - P_O H_E \rangle$.

Proof. This is straightforward. □

The following is the main result in this section, which provides an explicit finite generator of the 1-ideal I as a

$\mathbb{Z}[x_1, \ldots, x_r]$-module:

Theorem 4.8. As a $\mathbb{Z}[x_1, \ldots, x_r]$-module, the 1-ideal I is generated by the following four homogeneous elements:

$$\begin{pmatrix} P\dfrac{P_EH_O - P_OH_E}{2} \\ Q\dfrac{-P_EH_O - P_OH_E}{2} \end{pmatrix}, \begin{pmatrix} PP_E \\ -QP_E \end{pmatrix}, \begin{pmatrix} PP_O \\ QP_O \end{pmatrix}, \begin{pmatrix} 0 \\ QM_p \end{pmatrix}.$$

Proof. It is straightforward to check that the above four elements are certainly in the 1-ideal I.

Let $\begin{pmatrix} PA \\ QB \end{pmatrix} \in I$. Then (A, B) is in the set S of Lemma 4.6. We set $(A', B') := f(A, B)$ for simplicity. By combining Lemma 4.6 and Lemma 4.7, we have

$$P_EA' + P_OB' = F(2P_E) + F'(2P_O) + G(P_EH_O - P_OH_E)$$

or equivalently,

$$P_E(A' - 2F - GH_O) = P_O(-B' + 2F' - GH_E)$$

for some $F, F', G \in \mathbb{Z}[x_1, \ldots, x_r]$. Since P_E and P_O are coprime, one can take a polynomial $H \in \mathbb{Z}[x_1, \ldots, x_r]$ so that

$$A' - 2F - GH_O = P_OH.$$

Then we have

$$A' = 2F + GH_O + P_OH, \quad B' = 2F' - GH_E - P_EH.$$

Therefore, by Lemma 4.6 we have

$$\begin{aligned} A &= \frac{P_EA' + P_OB'}{2} \\ &= \frac{1}{2}\Big(P_E(2F + GH_O + P_OH) + P_O(2F' - GH_E - P_EH)\Big) \\ &= G\frac{P_EH_O - P_OH_E}{2} + FP_E + F'P_O, \\ B &= \frac{-P_EA' + P_OB'}{2} \\ &= \frac{1}{2}\Big(-P_E(2F + GH_O + P_OH) + P_O(2F' - GH_E - P_EH)\Big) \\ &= G\frac{-P_EH_O - P_OH_E}{2} + (-F)P_E + F'P_O - P_EP_OH \\ &= G\frac{-P_EH_O - P_OH_E}{2} + (-F)P_E + F'P_O - HM_p \ \text{(by Remark 4.5)}. \end{aligned}$$

In conclusion we have

$$\begin{aligned} \begin{pmatrix} PA \\ QB \end{pmatrix} &= \begin{pmatrix} P\Big(G\dfrac{P_EH_O - P_OH_E}{2} + FP_E + F'P_O\Big) \\ Q\Big(G\cdot\dfrac{-P_EH_O - P_OH_E}{2} + (-F)P_E + F'P_O - HM_p\Big) \end{pmatrix} \\ &= G\begin{pmatrix} P\dfrac{P_EH_O - P_OH_E}{2} \\ Q\dfrac{-P_EH_O - P_OH_E}{2} \end{pmatrix} + F\begin{pmatrix} PP_E \\ -QP_E \end{pmatrix} + F'\begin{pmatrix} PP_O \\ QP_O \end{pmatrix} + (-H)\begin{pmatrix} 0 \\ QM_p \end{pmatrix}. \end{aligned}$$

The proof is now complete. □

Following Theorem 4.8 we set

$$X := \begin{pmatrix} P\dfrac{P_E H_O - P_O H_E}{2} \\ Q\dfrac{-P_E H_O - P_O H_E}{2} \end{pmatrix},\ Y := \begin{pmatrix} PP_E \\ -QP_E \end{pmatrix},\ Z := \begin{pmatrix} PP_O \\ QP_O \end{pmatrix},\ W := \begin{pmatrix} 0 \\ QM_p \end{pmatrix}.$$

Remark 4.9. (i) The generator $\{X, Y, Z, W\}$ is redundant in general. In addition, the generator is not a free base even in the case that it is irredundant. This is because two equalities $2X = H_O Y - H_E Z$, $2W = -P_O Y + P_E Z$ hold.

(ii) Y and Z are linearly independent over $\mathbb{Z}[x_1, \ldots, x_r]$.

(iii) We have

$$\deg Y = \deg P + |E|, \quad \deg Z = \deg Q + |O|, \quad \deg W = 2n$$

5 Classification of 1-ideals

In this section we consider isomorphisms of 1-ideals. Let $\mathcal{G}$ and $\mathcal{G}'$ be GKMgraphs of type (r, n). We also fix $p, q \in \mathcal{V}$ and $p', q' \in \mathcal{V}'$. We assume that both of vertex pairs (p, q) and (p', q') are adjacent pairs. We identify $I_{pq} = I$ and $I_{p'q'} = I'$ as in Section 4. We also assume that there exists an isomorphism $\varphi : I \to I'$.

Remark 5.1. Note that the 1-ideal I does not have multiplicative unit. Thus I is a non-unital (but associative and commutative) graded ring. When we speak of an isomorphism of 1-ideals, we understand that it is a bijection preserving the degree, the addition, the multiplication, and the $\mathbb{Z}[x_1, \ldots, x_r]$-action.

Remark 5.2. In Section 4 we have used the symbols

$$p, q, P, Q, M_p, M_q, P_E, P_O, I, X, Y, Z, W,$$

and so on. The corresponding symbols for $\mathcal{G}'$ are written by putting primes:

$$p', q', P', Q', M_{p'}, M_{q'}, P'_E, P'_O, I', X', Y', Z', W', \ldots.$$

Our main purpose in this section is to show that $M_{p'} = \pm M_p$. We first explain our strategy for proving the equality.

In Theorem 4.8 a $\mathbb{Z}[x_1, \ldots, x_r]$-module generator of I is obtained. It is not difficult to write down the structure constants with respect to multiplication of I. However, there are ten relations in total (X^2, Y^2, Z^2, W^2, $XY, \ldots$) and it is somewhat complicated to deal with all of these relations. In fact, thanks to the following easy lemma, one may ignore most of these relations:

Corollary 5.3. The set $2I := \{2f | f \in I\}$ is contained in the direct sum $\mathbb{Z}[x_1, \ldots, x_r]Y \oplus \mathbb{Z}[x_1, \ldots, x_r]Z$.

Proof. This immediately follows from Theorem 5.8 and the equalities in Remark 4.9 (i). □

Remark 5.4. In Corollary 5.3, the elements Y, Z are not in $2I$.

Notice that a homomorphism $I \to I'$ induces a homomorphism $2I \to 2I'$. In addition, the former is an isomorphism if and only if the later is. By combining this fact and Corollary 5.3, it turns out that one may ignore all relations containing at least one of X, W.

In conclusion, the following lemma completes all relations we need:

Lemma 5.5. We have the following three relations:

(1) $2Y^2 = P_E(P - Q)Y + P_E^2 \dfrac{P + Q}{P_O} Z.$

(2) $2Z^2 = P_O^2 \dfrac{P - Q}{P_E} Y + P_O(P + Q)Z.$

(3) $2YZ = P_O(P + Q)Y + P_E(P - Q)Z.$

Proof. Easily verified by the definition of Y and Z. □

We next show that I and I' have "the same size" (see Corollary 5.8 below for precise statement).

We begin by the following general lemma:

Lemma 5.6. Let R be a non-zero graded ring such that the map $R \to R$, $r \mapsto 2r$ is injective. Let M be a graded R-module such that the map $M \to M$, $m \mapsto rm$ is injective for any $r \in R \setminus \{0\}$.

Assume that two homogeneous elements u, v of M are linearly independent over R, and $2M := \{2m | m \in M\}$ is contained in the direct sum $Ru \oplus Rv$. Then the set $\{\deg u, \deg v\}$ is independent of choice of u, v.

Proof. We take another u', v'. Without loss of generality, we may assume that $\deg u \geq \deg v$ and $\deg u' \geq \deg v'$. By assumption one can set $2u = au' + bv'$, $2v = cu' + dv'$ for some $a, b, c, d \in \mathbb{Z}[x_1, \ldots, x_r]$.

We first show that $\deg v = \deg v'$. By symmetry, it is enough to show that $\deg v \geq \deg v'$. If $d \neq 0$, the inequality is trivial since $dv' \neq 0$. If $d = 0$, we have $c \neq 0$ since $2v \neq 0$ by assumption. Thus $\deg v \geq \deg u' \geq \deg v'$ as desired.

We next show that $\deg u = \deg u'$. By symmetry, it is enough to show that $\deg u \geq \deg u'$.

Assume that a $= c = 0$. Then we have

$$2du - 2bv = d(2u) - b(2v) = d(bv') - b(dv') = 0.$$

Since u and v are linearly independent over R, we have $2b = 2d = 0$. The assumption on R implies $b = d = 0$. This is a contradiction. Thus at least one of a, c is a non-zero element of R.

If $a \neq 0$, the desired inequality is trivial. If $a = 0$, we have $c \neq 0$. Thus we have

$$\deg u \geq \deg v' = \deg v \geq \deg u'.$$

The proof is now complete. □

Corollary 5.7. We have

$$\{\deg Y, \deg Z\} = \{\deg Y', \deg Z'\}.$$

Proof. This follows from Corollary 5.3 and Lemma 5.6 □

Corollary 5.8. We have

$$\deg P = \deg P', \deg M_p = \deg M_{p'}, \{|E|, |O|\} = \{|E'|, |O'|\}.$$

Proof. We have

$$\begin{aligned} 2n + \deg P &= (\deg P + \deg M_p) + \deg P \\ &= (\deg P + \deg P_E) + (\deg P + \deg P_O) \quad \text{(Remark 4.5)} \\ &= \deg Y + \deg Z \\ &= \deg Y' + \deg Z' \quad \text{(Corollary 5.7)} \\ &= 2n + \deg P' \quad \text{(by the same calculation).} \end{aligned}$$

Thus the equality $\deg P = \deg P'$ follows. Since

$$\deg M_p = 2n - \deg P = 2n - \deg P' = \deg M_{p'},$$

the second equality holds. Finally, the last equality follows from $\deg P = \deg P'$ and Corollary 5.7. □

By Corollary 5.3, one can set

$$\varphi(2Y) = AY' + BZ', \quad \varphi(2Z) = CY' + DZ'$$

for some $A, B, C, D \in \mathbb{Z}[x_1, \ldots, x_r]$.

Lemma 5.9. The polynomial $AD - BC$ is a non-zero integer.

Proof. By Corollary 5.3, one can set

$$\varphi^{-1}(2Y') = A'Y + B'Z, \quad \varphi^{-1}(2Z') = C'Y + D'Z$$

for some $A', B', C', D' \in \mathbb{Z}[x_1, \ldots, x_r]$. Then we easily see that

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

By taking the determinant of both sides, we obtain the lemma. □

Now we calculate $\varphi(8Y^2)$ in the following two ways:

First, by Lemma 5.5 (1) we have

$$\begin{aligned}
\varphi(8Y^2) &= 4\varphi(2Y^2) \\
&= 4\varphi\left((P - Q)P_E Y + P_E^2 \frac{P + Q}{P_O} Z \right) \\
&= 2(P - Q)P_E \varphi(2Y) + 2P_E^2 \frac{P + Q}{P_O} \varphi(2Z) \\
&= 2(P - Q)P_E(AY' + BZ') + 2P_E^2 \frac{P + Q}{P_O} (CY' + DZ').
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\varphi(8Y^2) &= 2(\varphi(2Y))^2 \\
&= 2(AY' + BZ')^2 \\
&= A^2(2(Y')^2) + 2AB(2Y'Z') + B^2(2(Z')^2) \\
&= A^2\left((P' - Q')P_{E'} Y' + P_{E'}^2 \frac{P' + Q'}{P_{O'}} Z' \right) \\
&\quad + 2AB\left((P' + Q')P_{O'} Y' + (P' - Q')P_{E'} Z' \right) \\
&\quad + B^2\left(P_{O'}^2 \frac{P' - Q'}{P_{E'}} Y' + (P' + Q')P_{O'} Z' \right).
\end{aligned}$$

Recall that Y' and Z' are linearly independent over $\mathbb{Z}[x_1, \ldots, x_r]$. Thus, by comparing the coefficients we have two equalities on polynomials appearing above. Similarly Lemma 5.5 (2),(3) yield yet another four relations. Thus we obtain six relations in total. The results are the following:

(R 1) $2P_{E'}\left(AM_p(P - Q) + CP_E^2(P + Q) \right)$

$$= P_O\left(A^2P_{E'}^2 (P' - Q') + 2ABM_{p'} (P' + Q') + B^2P_{O'}^2 (P' - Q') \right)$$

(R 2) $2P_{O'}\left(BM_p(P - Q) + DP_E^2(P + Q) \right)$

$$= P_O\left(A^2P_{E'}^2 (P' + Q') + 2ABM_{p'} (P' - Q') + B^2P_{O'}^2 (P' + Q') \right)$$

(R 3) $2P_{E'}\left(AP_O^2(P - Q) + CM_p(P + Q) \right)$

$$= P_E\left(C^2P_{E'}^2 (P' - Q') + 2CDM_{p'} (P' + Q') + D^2P_{O'}^2 (P' - Q') \right)$$

(R 4) $2P_{O'}\Big(BP_O^2(P-Q)+DM_p(P+Q)\Big)$

$$=P_E\Big(C^2P_{E'}^2(P'+Q')+2CDM_{p'}(P'-Q')+D^2P_{O'}^2(P'+Q')\Big)$$

(R 5) $2P_{E'}\Big(AP_O(P+Q)+CP_E(P-Q)\Big)$

$$=ACP_{E'}^2(P'-Q')+(AD+BC)M_{p'}(P'+Q')+BDP_{O'}^2(P'-Q')$$

(R 6) $2P_{O'}\Big(BP_O(P+Q)+DP_E(P-Q)\Big)$

$$=ACP_{E'}^2(P'+Q')+(AD+BC)M_{p'}(P'-Q')+BDP_{O'}^2(P'+Q').$$

Lemma 5.10. (1) $2AP_{E'}$ and $2BP_{O'}$ are both divisible by P_E.
(2) $2CP_{E'}$ and $2DP_{O'}$ are both divisible by P_O.

Proof. Note that by Corollary 4.4 and Remark 4.5 both of the left hand sides of (R 1) and (R 2) are divisible by P_E^2. Thus

$$\begin{aligned}(\text{RHS of (R 1)})+(\text{RHS of (R 2)}) &= P_O(2A^2P'P_{E'}^2+4ABP'M_{p'}+2B^2P'P_{O'}^2)\\ &= 2P_OP'(AP_{E'}+BP_{O'})^2\end{aligned}$$

is also divisible by P_E^2. We claim that $AP_{E'}+BP_{O'}$ is divisible by P_E. In the case that $P_E=1$, the claim is trivial. The claim is also true in the case that $P_E\neq 1$ because P_E and P_O are coprime and P' is square free. Since

$$\begin{aligned}(\text{RHS of (R 1)})-(\text{RHS of (R 2)}) &= P_O(-2A^2Q'P_{E'}^2+4ABQ'M_{p'}-2B^2Q'P_{O'}^2)\\ &= -2P_OQ'(AP_{E'}-BP_{O'})^2,\end{aligned}$$

the same argument shows that $AP_{E'}-BP_{O'}$ is also divisible by P_E. Now the equalities

$$2AP_{E'}=(AP_{E'}+BP_{O'})+(AP_{E'}-BP_{O'})$$

and

$$2BP_{O'}=(AP_{E'}+BP_{O'})-(AP_{E'}-BP_{O'})$$

complete the proof of (1). The same argument using (R 3),(R 4) proves (2). □

We finally prove the equality $M_{p'}=\pm M_p$.

Theorem 5.11. $M_{p'}=\varepsilon_pM_p$ for some εp $\in\{\pm 1\}$.

Proof. By Corollary 5.8 we can divide the proof of Theorem 5.11 into the following two cases:

Case 1 $|E|=|E'|$ and $|O|=|O'|$

Case 2 $|E|=|O'|$ and $|O|=|E'|$.

We first consider Case 1. In this case we have deg $Y=$ deg Y', deg $Z=$ deg Z'.
Thus both A and D are integers by degree reason. By combining this fact and Lemma 5.9, one finds that BC is also an integer. Therefore we can divide Case 1 into the following two cases:

Case 1-1 $BC=0$.

Case 1-2 Both of B and C are non-zero integers.

We first deal a part of Case 1-1, that is, the case that $B=0$. In this case we have $2\,\varphi(Y)=AY'$. Since the components of Y' are not divisible by 2, the integer A must be divisible by 2. Then the equality $Y=(\mathrm{A}/2)\varphi^{-1}(Y')$ implies $A=2$ or -2 since the components of Y are not divisible by 2. We set $A=2\varepsilon$ with $\varepsilon\in\{\pm 1\}$.

Then we have $\varphi(Y) = \varepsilon Y'$. Since A is a non-zero integer,
Lemma 5.10 (1) implies that $P_{E'}$ is divisible by P_E. By this divisibility, the equality $|E| = |E'|$, and the primitivity of P_E and $P_{E'}$, we have $P_{E'} = \eta P_E$ for some $\eta \in \{\pm 1\}$.

We next prove $P_{O'} = \pm P_O$. By evaluating $A = 2\varepsilon$, $B = 0$ and $P_{E'} = \eta P_E$ at the relations (R 1)¿(R 6), we have

(R 1)' $2\varepsilon P_O(P - Q) + CP_E(P + Q) = 2\eta P_O(P' - Q')$
(R 2)' $DP_{O'}(P + Q) = 2P_O(P' + Q')$
(R 3)' $2\eta\left(2\varepsilon(P - Q)P_O^2 + CM_p(P + Q)\right)$
$$= C^2P_E^2(P' - Q') + 2CDM_{p'}(P' + Q') + D^2P_{O'}^2(P' - Q')$$
(R 4)' $2DP_OP_{O'}(P + Q) = C^2P_E^2(P' + Q') + 2CDM_{p'}(P' - Q') + D^2P_{O'}^2(P' + Q')$
(R 5)' $2\varepsilon P_O(P + Q) + CP_E(P - Q) = \varepsilon\eta CP_E(P' - Q') + \varepsilon DP_{O'}(P' + Q')$
(R 6)' $DP_{O'}(P - Q) = \varepsilon CP_E(P' + Q') + \varepsilon\eta DP_{O'}(P' - Q')$

If $P + Q = 0$, (R 2)' implies that $P' + Q' = 0$. Thus, by (R 6)' we have $P - Q = \varepsilon\eta(P' - Q')$ (notice that $D \neq 0$ since $AD = AD - BC \neq 0$). These three equalities deduce that $P = \varepsilon\eta P'$, $Q = \varepsilon\eta Q'$. By evaluating these equalities at (R 4)', we have $2(P' - Q')M_{p'}CD = 0$. Therefore $C = 0$ since $P' - Q' \neq 0$, $D \neq 0$ as noticed above. Then (R 3)' becomes $4\varepsilon\eta P_O^2(P - Q) = D^2P_{O'}^2(P' - Q')$. From this equality, one easily deduces that $D = \pm 2$, $P_{O'} = \pm P_O$.

If $P + Q \neq 0$, (R 2)' implies that $D = \dfrac{2P_O(P' + Q')}{P_{O'}(P + Q)}$. By evaluating this at (R 6)', we have

$$2P_O(P - Q) = \varepsilon CP_E(P + Q) + 2\varepsilon\eta P_O(P' - Q').$$

From this equality and (R 1)' we immediately have $C = 0$. Then, (R 4)' becomes $(P + Q)^2 = (P' + Q')^2$ and thus $P + Q = \sigma(P' + Q')$ for some $\sigma \in \{\pm 1\}$. Therefore, by (R 2)' we have $\sigma DP_{O'} = 2P_O$. This implies that $D = \pm 2$, $P_{O'} = P_O$.

The proof in the case that $B = 0$ is now complete.

In the case that $C = 0$, we have $\varphi(2Z) = DZ'$ and $\deg Z = \deg Z'$. The same argument using (R 1)-(R 6) shows that $B = 0$. Thus the proof is reduced to the previous case. The proof in Case 1-1 is now complete.

We next consider Case 1-2. We first show that $AD = 0$ by contradiction. Assume that $AD \neq 0$. Then all of A, B, C, D are non-zero integers. In particular, Lemma 5.10 shows that both $P_{E'}$ and $P_{O'}$ are divisible by P_EP_O. This contradicts the GKM-condition because at least one of P_E and P_O is of degree ≥ 2. Thus we have $AD = 0$ as desired.

If $A = 0$, we have $\varphi(2Y) = BZ'$, $\deg Y = \deg Z'$. An argument similar to Case 1-1 shows that $B = 2\varepsilon$ for some $\varepsilon \in \{\pm 1\}$. By Lemma 5.10 (1) $P_{O'}$ is divisible by P_E. On the other hand we have $|E| = |O'|$ because

$$\deg P + |E| = \deg Y = \deg Z' = \deg Q + |O'| = \deg P + |O'|.$$

Thus we have $P_{O'} = \eta P_E$ for some $\eta \in \{\pm 1\}$. Now, relations (R 1)-(R 6) show that $C = \pm 2$, $P_{E'} = \pm P_O$.

If $D = 0$, we have $\varphi(2Z) = CY'$, $\deg Z = \deg Y'$. By an argument similar to the case that $A = 0$, we have $|O| = |E'|$. Thus we obtain $P_{E'} = \eta P_O$ for some $\eta \in \{\pm 1\}$. Now relations (R 1)-(R 6) shows that $A = 0$. Thus the proof is reduced to the previous case.

The proof in Case 1 is now complete.

The proof in Case 1 also works in Case 2. This completes the proof of Theorem 6.10. □

6 Proof of main theorem

In this last section we prove our main theorem in this paper. Let $\mathcal{G}$ and $\mathcal{G}'$ be GKM-graphs of type (r, n) and (r, n') respectively. We fix a graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebra isomorphism $\varphi\colon H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$. Note that we have $n = n'$ by Corollary 2.7.

By Theorem 2.6, there exists a unique bijection $\varphi_{\mathcal{V}}\colon \mathcal{V}' \to \mathcal{V}$ satisfying

$$\varphi(\tau_{\varphi_\mathcal{V}(p')}) = \varepsilon_{p'}\tau_{p'}$$

for some $\varepsilon_{p'} \in \{\pm 1\}$.

Lemma 6.1. $\varphi_\mathcal{V}(p')$ and $\varphi_\mathcal{V}(q')$ are adjacent if and only if p' and q' are so.

Proof. We set $p := \varphi_\mathcal{V}(p')$, $q := \varphi_\mathcal{V}(q')$ for simplicity. By symmetry, it is enough to show that if p and q are adjacent, then p' and q' are so.

We use the notation of Section 5. By the definition of Y and Z, we have

$$P_O Y + P_E Z = 2\,\tau_p, \quad P_O Y - P_E Z = 2\varepsilon\tau_q$$

for some $\varepsilon \in \{\pm 1\}$. Thus we have

$$P_O Y = \tau_p + \varepsilon\tau_q, \quad P_E Z = \tau_p - \varepsilon\tau_q.$$

These equalities and Theorem 2.6 imply that

$$P_O\varphi(Y) = \varepsilon_{p'}\tau_{p'} + \varepsilon\varepsilon_{q'}\tau_{q'}, \quad P_E\varphi(Z) = \varepsilon_{p'}\tau_{p'} - \varepsilon\varepsilon_{q'}\tau_{q'}.$$

In particular both $\varphi(Y)$ and $\varphi(Z)$ are in $I_{p'q'}$. Since at least one of deg Y and deg Z is less than $2n$, Lemma 3.4 shows that p' and q' are adjacent. □

Corollary 6.2. We have $\varphi(I_{\varphi_\mathcal{V}(p')\varphi_\mathcal{V}(q')}) = I_{p'q'}$ for all $p', q' \in \mathcal{V}'$.

Proof. We use the notation of Lemma 6.1. If p and q are not adjacent, the equality is trivial by Lemma 3.4 and Lemma 6.1. Therefore, we may assume that p and q are adjacent. In this case, we already proved in the proof of Lemma 6.1 that both $\varphi(Y)$ and $\varphi(Z)$ are in $I_{p'q'}$. Then, Remark 4.9 (i) and Theorem 4.8 imply that $\varphi(I_{pq}) \subset I_{p'q'}$. By symmetry we also have $\varphi^{-1}(I_{p'q'}) \subset I_{pq}$. The proof is now complete. □

We are now in the position to prove Theorem 1.1 :

Proof of Theorem 1.1. The "if" part is clear. To show the "only if" part, take an isomorphism $\varphi : H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$ of graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebras. By Theorem 2.6, there exists a bijection $\varphi_\mathcal{V}\colon \mathcal{V}' \to \mathcal{V}$ so that $\varphi(\tau_{\varphi_\mathcal{V}(p')}) = \varepsilon_{p'}\tau_{p'}$ for some $\varepsilon_{p'} \in \{\pm 1\}$.

Assume that p' and q' are adjacent. By Corollary 6.2 we have an isomorphism $\varphi|I_{\varphi_\mathcal{V}(p')\varphi_\mathcal{V}(q')} : I_{\varphi_\mathcal{V}(p')\varphi_\mathcal{V}(q')} \to I_{p'q'}$. Therefore, Theorem 5.11 shows that $M_{p'} = \sigma_{p'}M_{\varphi_\mathcal{V}(p')}$ for some $\sigma_{p'} \in \{\pm 1\}$.

The proof of Theorem 1.1 is now complete. □

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Rigidity of GKM graphs via 1-ideals

YAMANAKA Hitoshi

Franz and the author previously proved equivariant cohoomological rigidity for abstract GKM graphs. The present paper gives the author's original proof based on the notion of a 1-ideal.